Projective Geometry

Milivoje Lukić

Contents

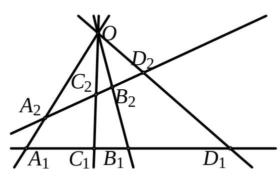
1 Cross Ratio. Harmonic Conjugates. Perspectivity. Projectivity

Definition 1. *Let A, B, C, and D be colinear points. The* cross ratio *of the pairs of points (A,B) and (C,D) is*

$$\mathscr{R}(A,B;C,D)=\frac{\overrightarrow{AC}}{\overrightarrow{CB}}:\frac{\overrightarrow{AD}}{\overrightarrow{DB}}. \tag{1}$$

Let a, b, c, d be four concurrent lines. For the given lines p_1 and p_2 let us denote $A_i=a\cap p_i$, $B_i=b\cap p_i$, $C_i=c\cap p_i$, $D_i=d\cap p_i$, for $i=1,2$. Then

$$\mathscr{R}(A_1,B_1;C_1,D_1)=\mathscr{R}(A_2,B_2;C_2,D_2). \tag{2}$$

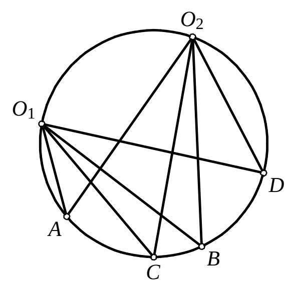


Thus it is meaningful to define the cross ratio of the pairs of concurrent points as

$$\mathscr{R}(a,b;c,d)=\mathscr{R}(A_1,B_1;C_1,D_1). \tag{3}$$

Assume that points O_1, O_2, A, B, C, D belong to a circle. Then

$$\begin{aligned}&\mathscr{R}(O_1A,O_1B;O_1C,O_1D)\\ =\ &\mathscr{R}(O_2A,O_2B;O_2C,O_2D).\end{aligned} \tag{4}$$

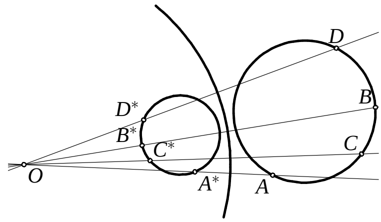


Hence it is meaningful to define the cross-ratio for cocyclic points as

$$\mathscr{R}(A,B;C,D)=\mathscr{R}(O_1A,O_1B;O_1C,O_1D). \tag{5}$$

Assume that the points A, B, C, D are colinear or cocyclic. Let an inversion with center O maps A, B, C, D into A^*, B^*, C^*, D^*. Then

$$\mathscr{R}(A,B;C,D) = \mathscr{R}(A^*,B^*;C^*,D^*). \qquad (6)$$



Definition 2. *Assume that A, B, C, and D are cocyclic or colinear points. Pairs of points (A,B) and (C,D) are* harmonic conjugates *if $\mathscr{R}(A,B;C,D) = -1$. We also write $\mathscr{H}(A,B;C,D)$ when we want to say that (A,B) and (C,D) are harmonic conjugates to each other.*

Definition 3. *Let each of l_1 and l_2 be either line or circle.* Perspectivity *with respect to the point S $\overset{S}{\barwedge}$, is the mapping of $l_1 \to l_2$, such that*

(i) If either l_1 or l_2 is a circle than it contains S;

(ii) every point $A_1 \in l_1$ is mapped to the point $A_2 = OA_1 \cap l_2$.

According to the previous statements perspectivity preserves the cross ratio and hence the harmonic conjugates.

Definition 4. *Let each of l_1 and l_2 be either line or circle.* Projectivity *is any mapping from l_1 to l_2 that can be represented as a finite composition of perspectivities.*

Theorem 1. *Assume that the points A, B, C, D_1, and D_2 are either colinear or cocyclic. If the equation $\mathscr{R}(A,B;C,D_1) = \mathscr{R}(A,B;C,D_2)$ is satisfied, then $D_1 = D_2$. In other words, a projectivity with three fixed points is the identity.*

Theorem 2. *If the points A, B, C, D are mutually discjoint and $\mathscr{R}(A,B;C,D) = \mathscr{R}(B,A;C,D)$ then $\mathscr{H}(A,B;C,D)$.*

2 Desargue's Theorem

The triangles $A_1B_1C_1$ and $A_2B_2C_2$ are *perspective with respect to a center* if the lines A_1A_2, B_1B_2, and C_1C_2 are concurrent. They are *perspective with respect to an axis* if the points $K = B_1C_1 \cap B_2C_2$, $L = A_1C_1 \cap A_2C_2$, $M = A_1B_1 \cap A_2B_2$ are colinear.

Theorem 3 (Desargue). *Two triangles are perspective with respect to a center if and only if they are perspective with respect to a point.*

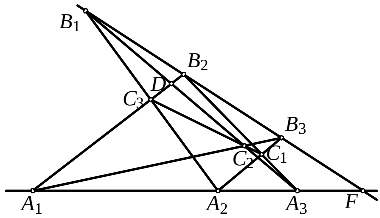
3 Theorems of Pappus and Pascal

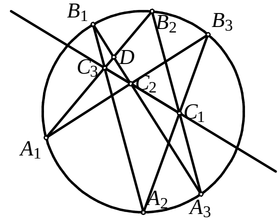
Theorem 4 (Pappus). *The points A_1, A_2, A_3 belong to the line a, and the points B_1, B_2, B_3 belong to the line b. Assume that $A_1B_2 \cap A_2B_1 = C_3$, $A_1B_3 \cap A_3B_1 = C_2$, $A_2B_3 \cap A_3B_2 = C_1$. Then C_1, C_2, C_3 are colinear.*

Proof. Denote $C_2' = C_1C_3 \cap A_3B_1$, $D = A_1B_2 \cap A_3B_1$, $E = A_2B_1 \cap A_3B_2$, $F = a \cap b$. Our goal is to prove that the points C_2 and C_2' are identical. Consider the sequence of projectivities:

$$A_3B_1DC_2 \overset{A_1}{\barwedge} FB_1B_2B_3 \overset{A_2}{\barwedge} A_3EB_2C_1 \overset{C_3}{\barwedge} A_3B_1DC_2'.$$

We have got the projective transformation of the line A_3B_1 that fixes the points A_3, B_1, D, and maps C_2 to C_2'. Since the projective mapping with three fixed points is the identity we have $C_2 = C_2'$. □





Theorem 5 (Pascal). *Assume that the points A_1, A_2, A_3, B_1, B_2, B_3 belong to a circle. The point in intersections of A_1B_2 with A_2B_1, A_1B_3 with A_3B_1, A_2B_3 with A_3B_2 lie on a line.*

Proof. The points C_2', D, and E as in the proof of the Pappus theorem. Consider the sequence of perspectivities

$$A_3B_1DC_2 \overset{A_1}{\barwedge} A_3B_1B_2B_3 \overset{A_2}{\barwedge} A_3EB_2C_1 \overset{C_3}{\barwedge} A_3B_1DC_2'.$$

In the same way as above we conclude that $C_2 = C_2'$. $\square$

4 Pole. Polar. Theorems of Brianchon and Brokard

Definition 5. *Given a circle $k(O,r)$, let A^* be the image of the point $A \neq O$ under the inversion with respect to k. The line a passing through A^* and perpendicular to OA is called the* polar *of A with respect to k. Conversely A is called the* pole *of a with respect to k.*

Theorem 6. *Given a circle $k(O,r)$, let and a and b be the polars of A and B with respect to k. The $A \in b$ if and only if $B \in a$.*

Proof. $A \in b$ if and only if $\angle AB^*O = 90^\circ$. Analogously $B \in a$ if and only if $\angle BA^*O = 90^\circ$, and it reamins to notice that according to the basic properties of inversion we have $\angle AB^*O = \angle BA^*O$. $\square$

Definition 6. *Points A and B are called* conjugated *with respect to the circle k if one of them lies on a polar of the other.*

Theorem 7. *If the line determined by two conjugated points A and B intersects $k(O,r)$ at C and D, then $\mathscr{H}(A,B;C,D)$. Conversely if $\mathscr{H}(A,B;C,D)$, where $C,D \in k$ then A and B are conjugated with respect to k.*

Proof. Let C_1 and D_1 be the intersection points of OA with k. Since the inversion preserves the cross-ratio and $\mathscr{R}(C_1,D_1;A,A^*) = \mathscr{R}(C_1,D_1;A^*,A)$ we have

$$\mathscr{H}(C_1,D_1;A,A^*). \tag{7}$$

Let p be the line that contains A and intersects k at C and D. Let $E = CC_1 \cap DD_1$, $F = CD_1 \cap DC_1$. Since C_1D_1 is the diameter of k we have $C_1F \perp D_1E$ and $D_1F \perp C_1E$, hence F is the orthocenter of the triangle C_1D_1E. Let $B = EF \cap CD$ and $\overline{A}^* = EF \cap C_1D_1$. Since

$$C_1D_1A\overline{A}^* \overset{E}{\barwedge} CDAB \overset{F}{\barwedge} D_1C_1A\overline{A}^*$$

have $\mathscr{H}(C_1,D_1;A,\overline{A}^*)$ and $\mathscr{H}(C,D;A,B)$. (7) now implies two facts:

1° From $\mathscr{H}(C_1,D_1;A,\overline{A}^*)$ and $\mathscr{H}(C_1,D_1;A,A^*)$ we get $A^* = \overline{A}^*$, hence $A^* \in EF$. However, since $EF \perp C_1D_1$, the line $EF = a$ is the polar of A.

2° For the point B which belongs to the polar of A we have $\mathscr{H}(C,D;A,B)$. This completes the proof. $\square$

Theorem 8 (Brianchon's theorem). *Assume that the hexagon $A_1A_2A_3A_4A_5A_6$ is circumscribed about the circle k. The lines A_1A_4, A_2A_5, and A_3A_6 intersect at a point.*

Proof. We will use the convention in which the points will be denoted by capital latin letters, and their repsective polars with the corresponding lowercase letters.

Denote by M_i, $i = 1,2,\dots,6$, the points of tangency of A_iA_{i+1} with k. Since $m_i = A_iA_{i+1}$, we have $M_i \in a_i$, $M_i \in a_{i+1}$, hence $a_i = M_{i-1}M_i$.

Let $b_j = A_jA_{j+3}$, $j = 1,2,3$. Then $B_j = a_j \cap a_{j+3} = M_{j-1}M_j \cap M_{j+3}M_{j+4}$. We have to prove that there exists a point P such that $P \in b_1, b_2, b_3$, or analogously, that there is a line p such that $B_1, B_2, B_3 \in p$. In other words we have to prove that the points B_1, B_2, B_3 are colinear. However this immediately follows from the Pascal's theorem applied to $M_1M_3M_5M_4M_6M_2$. $\square$

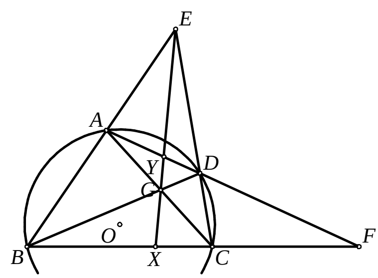
From the previous proof we see that the Brianchon's theorem is obtained from the Pascal's by replacing all the points with their polars and all lines by theirs poles.

Theorem 9 (Brokard). *The quadrilateral $ABCD$ is inscribed in the circle k with center O. Let $E = AB \cap CD$, $F = AD \cap BC$, $G = AC \cap BD$. Then O is the orthocenter of the triangle EFG.*

Proof. We will prove that EG is a polar of F. Let $X = EG \cap BC$ and $Y = EG \cap AD$. Then we also have

$$ADYF \overset{E}{\barwedge} BCXF \overset{G}{\barwedge} DAYF,$$

which implies the relations $\mathscr{H}(A,D;Y,F)$ and $\mathscr{H}(B,C;X,F)$. According to the properties of polar we have that the points X and Y lie on a polar of the point F, hence EG is a polar of the point F.



Since EG is a polar of F, we have $EG \perp OF$. Analogously we have $FG \perp OE$, thus O is the orthocenter of $\triangle EFG$. $\square$

5 Problems

1. Given a quadrilateral $ABCD$, let $P = AB \cap CD$, $Q = AD \cap BC$, $R = AC \cap PQ$, $S = BD \cap PQ$. Prove that $\mathscr{H}(P,Q;R,S)$.

2. Given a triangle ABC and a point M on BC, let N be the point of the line BC such that $\angle MAN = 90^\circ$. Prove that $\mathscr{H}(B,C;M,N)$ if and only if AM is the bisector of the angle $\angle BAC$.

3. Let A and B be two points and let C be the point of the line AB. Using just a ruler find a point D on the line AB such that $\mathscr{H}(A,B;C,D)$.

4. Let A, B, C be the diagonal points of the quadrilateral $PQRS$, or equivalently $A = PQ \cap RS$, $B = QR \cap SP$, $C = PR \cap QS$. If only the points A, B, C, S, are given using just a ruler construct the points P, Q, R.

5. Assume that the incircle of $\triangle ABC$ touches the sides BC, AC, and AB at D, E, and F. Let M be the point such that the circle k_1 incscibed in $\triangle BCM$ touches BC at D, and the sides BM and CM at P and Q. Prove that the lines EF, PQ, BC are concurrent.

6. Given a triangle ABC, let D and E be the points on BC such that $BD = DE = EC$. The line p intersects AB, AD, AE, AC at K, L, M, N, respectively. Prove that $KN \geq 3LM$.

7. The point M_1 belongs to the side AB of the quadrilateral $ABCD$. Let M_2 be the projection of M_1 to the line BC from D, M_3 projection of M_2 to CD from A, M_4 projection of M_3 to DA from B, M_5 projection of M_4 to AB from C, etc. Prove that $M_{13} = M_1$.

8. (butterfly theorem) Points M and N belong to the circle k. Let P be the midpoint of the chord MN, and let AB and CD (A and C are on the same side of MN) be arbitrary chords of k passing through P. Prove that lines AD and BC intersect MN at points that are equidistant from P.

9. Given a triangle ABC, let D and E be the points of the sides AB and AC respectively such that $DE \| BC$. Let P be an interior point of the triangle ADE. Assume that the lines BP and CP intersect DE at F and G respectively. The circumcircles of $\triangle PDG$ and $\triangle PFE$ intersect at P and Q. Prove that the points A, P, and Q are colinear.

10. (IMO 1997 shortlist) Let $A_1A_2A_3$ be a non-isosceles triangle with the incenter I. Let C_i, $i = 1$, 2, 3, be the smaller circle through I tangent to both A_iA_{i+1} and A_iA_{i+2} (summation of indeces is done modulus 3). Let B_i, $i = 1$, 2, 3, be the other intersection point of C_{i+1} and C_{i+2}. Prove that the circumcenters of the triangles A_1B_1I, A_2B_2I, A_3B_3I are colinear.

11. Given a triangle ABC and a point T, let P and Q be the feet of perpendiculars from T to the lines AB and AC, respectively. Let R and S be the feet of perpendiculars from A to TC and TB, respectively. Prove that the intersection of PR and QS belongs to BC.

12. Given a triangle ABC and a point M, a line passing through M intersects AB, BC, and CA at C_1, A_1, and B_1, respectively. The lines AM, BM, and CM intersect the circumcircle of $\triangle ABC$ repsectively at A_2, B_2, and C_2. Prove that the lines A_1A_2, B_1B_2, and C_1C_2 intersect in a point that belongs to the circumcircle of $\triangle ABC$.

13. Let P and Q isogonaly conjugated points and assume that $\triangle P_1P_2P_3$ and $\triangle Q_1Q_2Q_3$ are their pedal triangles, respectively. Let $X_1 = P_2Q_3 \cap P_3Q_2$, $X_2 = P_1Q_3 \cap P_3Q_1$, $X_3 = P_1Q_2 \cap P_2Q_1$. Prove that the points X_1, X_2, X_3 belong to the line PQ.

14. If the points A and M are conjugated with respect to k, then the circle with diameter AM is orthogonal to k.

15. From a point A in the exterior of a circle k two tangents AM and AN are drawn. Assume that K and L are two points of k such that A, K, L are colinear. Prove that MN bisects the segment PQ.

16. The point isogonaly conjugated to the centroid is called the *Lemuan* point. The lines connected the vertices with the Lemuan point are called *symmedians*. Assume that the tangents from B and C to the circumcircle Γ of $\triangle ABC$ intersect at the point P. Prove that AP is a symmedian of $\triangle ABC$.

17. Given a triangle ABC, assume that the incircle touches the sides BC, CA, AB at the points M, N, P, respectively. Prove that AM, BN, and CP intersect in a point.

18. Let $ABCD$ be a quadrilateral circumscribed about a circle. Let M, N, P, and Q be the points of tangency of the incircle with the sides AB, BC, CD, and DA respectively. Prove that the lines AC, BD, MP, and NQ intersect in a point.

19. Let $ABCD$ be a cyclic quadrilateral whose diagonals AC and BD intersect at O; extensions of the sides AB and CD at E; the tangents to the circumcircle from A and D at K; and the tangents to the circumcircle at B and C at L. Prove that the points E, K, O, and L lie on a line.

20. Let $ABCD$ be a cyclic quadrilateral. The lines AB and CD intersect at the point E, and the diagonals AC and BD at the point F. The circumcircle of the triangles $\triangle AFD$ and $\triangle BFC$ intersect again at H. Prove that $\angle EHF = 90^\circ$.

6 Solutions

1. Let $T = AC \cap BD$. Consider the sequence of the perspectivities
$$PQRS \overset{A}{\overline{\wedge}} BDTS \overset{C}{\overline{\wedge}} QPRS.$$
Since the perspectivity preserves the cross-ratio $\mathscr{R}(P,Q;R,S) = \mathscr{R}(Q,P;R,S)$ we obtain that $\mathscr{H}(P,Q;R,S)$.

2. Let $\alpha = \angle BAC$, $\beta = \angle CBA$, $\gamma = \angle ACB$ and $\varphi = \angle BAM$. Using the sine theorem on $\triangle ABM$ and $\triangle ACM$ we get
$$\frac{BM}{MC} = \frac{BM}{AM}\frac{AM}{CM} = \frac{\sin\varphi}{\sin\beta}\frac{\sin\gamma}{\sin(\alpha-\varphi)}.$$
Similarly using the sine theorem on $\triangle ABN$ and $\triangle ACN$ we get
$$\frac{BN}{NC} = \frac{BN}{AN}\frac{AN}{CN} = \frac{\sin(90^\circ-\varphi)}{\sin(180^\circ-\beta)}\frac{\sin\gamma}{\sin(90^\circ+\alpha-\varphi)}.$$
Combining the previous two equations we get
$$\frac{BM}{MC} : \frac{BN}{NC} = \frac{\tan\varphi}{\tan(\alpha-\varphi)}.$$
Hence, $|\mathscr{R}(B,C;M,N)| = 1$ is equivalent to $\tan\varphi = \tan(\alpha-\varphi)$, i.e. to $\varphi = \alpha/2$. Since $B \neq C$ and $M \neq N$, the relation $|\mathscr{R}(B,C;M,N)| = 1$ is equivalent to $\mathscr{R}(B,C;M,N) = -1$, and the statement is now shown.

3. The motivation is the problem 1. Choose a point K outside AB and point L on AK different from A and K. Let $M = BL \cap CK$ and $N = BK \cap AM$. Now let us construct a point D as $D = AB \cap LN$. From the problem 1 we indeed have $\mathscr{H}(A,B;C,D)$.

4. Let us denote $D = AS \cap BC$. According to the problem 1 we have $\mathscr{H}(R,S;A,D)$. Now we construct the point $D = AS \cap BC$. We have the points A, D, and S, hence according to the previous problem we can construct a point R such that $\mathscr{H}(A,D;S,R)$. Now we construct $P = BS \cap CR$ and $Q = CS \cap BR$, which solves the problem.

5. It is well known (and is easy to prove using Ceva's theorem) that the lines AD, BE, and CF intersect at a point G (called a Gergonne point of $\triangle ABC$) Let $X = BC \cap EF$. As in the problem 1 we have $\mathscr{H}(B,C;D,X)$. If we denote $X' = BC \cap PQ$ we analogously have $\mathscr{H}(B,C;D,X')$, hence $X = X'$.

6. Let us denote $x = KL$, $y = LM$, $z = MN$. We have to prove that $x+y+z \geq 3y$, or equivalently $x+z \geq 2y$. Since $\mathscr{R}(K,N;L,M) = \mathscr{R}(B,C;D,E)$, we have
$$\frac{x}{y+z} : \frac{x+y}{z} = \frac{\overrightarrow{KL}}{\overrightarrow{LN}} : \frac{\overrightarrow{KM}}{\overrightarrow{MN}} = \frac{\overrightarrow{BD}}{\overrightarrow{DC}} : \frac{\overrightarrow{BE}}{\overrightarrow{EC}} = \frac{1}{2} : \frac{1}{2},$$
implying $4xz = (x+y)(y+z)$.

   If it were $y > (x+z)/2$ we would have
$$x+y > \frac{3}{2}x + \frac{1}{2}z = 2\frac{1}{4}(x+x+x+z) \geq 2\sqrt[4]{xxxz},$$
and analogously $y+z > 2\sqrt[4]{xzzz}$ as well as $(x+y)(y+z) > 4xz$ which is a contradiction. Hence the assumption $y > (x+z)/2$ was false so we have $y \leq (x+z)/2$.

   Let us analyze the case of equality. If $y = (x+z)/2$, then $4xz = (x+y)(x+z) = (3x+z)(x+3z)/4$, which is equivalent to $(x-z)^2 = 0$. Hence the equality holds if $x = y = z$. We leave to the reader to prove that $x = y = z$ is satisfied if and only if $p \parallel BC$.

7. Let $E = AB \cap CD$, $F = AD \cap BC$. Consider the sequence of perspectivities

$$ABEM_1 \overset{D}{\barwedge} FBCM_2 \overset{A}{\barwedge} DECM_3 \overset{B}{\barwedge} DAFM_4 \overset{C}{\barwedge} EABM_5. \tag{8}$$

According to the conditions given in the problem this sequence of perspectivites has two be applied three more times to arrive to the point M_{13}. Notice that the given sequence of perspectivities maps A to E, E to B, and B to A. Clearly if we apply (8) three times the points A, B, and E will be fixed while M_1 will be mapped to M_{13}. Thus $M_1 = M_{13}$.

8. Let X' be the point symmetric to Y with respect to P. Notice that

$$\begin{aligned}
\mathscr{R}(M,N;X,P) &= \mathscr{R}(M,N;P,Y) \quad (\text{from } MNXP \overset{D}{\barwedge} MNAC \overset{B}{\barwedge} MNPY) \\
&= \mathscr{R}(N,M;P,X') \quad (\text{ the reflection with the center } P \text{ preserves} \\
&\qquad\qquad \text{the ratio, hence it preserves the cross-ratio}) \\
&= \frac{1}{\mathscr{R}(N,M;X',P)} = \mathscr{R}(M,N;X',P),
\end{aligned}$$

where the last equality follows from the basic properties of the cross ratio. It follows that $X = X'$.

9. Let $J = DQ \cap BP$, $K = EQ \cap CP$. If we prove that $JK \parallel DE$ this would imply that the triangles BDJ and CEK are perspective with the respect to a center, hence with repsect to an axis as well (according to Desargue's theorem) which immediately implies that A, P, Q are colinear (we encourage the reader to verify this fact).

   Now we will prove that $JK \| DE$. Let us denote $T = DE \cap PQ$. Applying the Menelaus theorem on the triangle DTQ and the line PF we get

$$\frac{\overrightarrow{DJ}}{\overrightarrow{JQ}} \frac{\overrightarrow{QP}}{\overrightarrow{PT}} \frac{\overrightarrow{TF}}{\overrightarrow{FD}} = -1.$$

   Similarly from the triangle ETQ and the line PG:

$$\frac{\overrightarrow{EK}}{\overrightarrow{KQ}} \frac{\overrightarrow{QP}}{\overrightarrow{PT}} \frac{\overrightarrow{TG}}{\overrightarrow{GE}} = -1.$$

   Dividing the last two equalities and using $DT \cdot TG = FT \cdot TE$ (T is on the radical axis of the circumcircles of $\triangle DPG$ and $\triangle FPE$), we get

$$\frac{\overrightarrow{DJ}}{\overrightarrow{JQ}} = \frac{\overrightarrow{EK}}{\overrightarrow{KQ}}.$$

   Thus $JK \parallel DE$, q.e.d.

10. Apply the inversion with the respect to I. We leave to the reader to draw the inverse picture. Notice that the condition that I is the incentar now reads that the circumcircles $A_i^*A_{i+1}^*I$ are of the same radii. Indeed if R is the radius of the circle of inversion and r the distance between I and XY then the radius of the circumcircle of $\triangle IX^*Y^*$ is equal to R^2/r. Now we use the following statement that is very easy to prove: "Let k_1, k_2, k_3 be three circles such that all pass through the same point I, but no two of them are mutually tangent. Then the centers of these circles are colinear if and only if there exists another common point $J \neq I$ of these three circles."

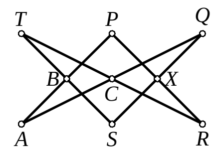
    In the inverse picture this transforms into proving that the lines $A_1^*B_1^*$, $A_2^*B_2^*$, and $A_3^*B_3^*$ intersect at a point.

In order to prove this it is enough to show that the corresponding sides of the triangles $A_1^*A_2^*A_3^*$ and $B_1^*B_2^*B_3^*$ are parallel (then these triangles would be perspective with respect to the infinitely far line). Afterwards the Desargue's theorem would imply that the triangles are perspective with respect to a center. Let P_i^* be the incenter of $A_{i+1}^*A_{i+2}^*I$, and let Q_i^* be the foot of the perpendicular from I to $P_{i+1}^*P_{i+2}^*$. It is easy to prove that

$$\overrightarrow{A_1^*A_2^*} = 2\overrightarrow{Q_1^*Q_2^*} = -\overrightarrow{P_1^*P_2^*}.$$

Also since the circles $A_i^*A_{i+1}^*I$ are of the same radii, we have $P_1^*P_2^* \parallel B_1^*B_2^*$, hence $A_1^*A_2^* \parallel B_1^*B_2^*$.

11. We will prove that the intersection X of PR and QS lies on the line BC. Notice that the points P, Q, R, S belong to the circle with center AT. Consider the six points A, S, R, T, P, Q that lie on a circle. Using Pascal's theorem with respect to the diagram



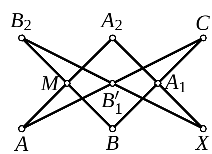
we get that the points B, C, and $X = PR \cap QS$ are colinear.

12. *First solution, using projective mappings.* Let $A_3 = AM \cap BC$ and $B_3 = BM \cap AC$. Let X be the other intersection point of the line A_1A_2 with the circumcircle k of $\triangle ABC$. Let X' be the other intersection point of the line B_1B_2 with k. Consider the sequence of perspectivities

$$ABCX \overset{A_2}{\barwedge} A_3BCA_1 \overset{M}{\barwedge} AB_3CB_1 \overset{B_2}{\barwedge} ABCX'$$

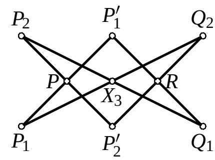
which has three fixed points A, B, C, hence $X = X'$. Analogously the line C_1C_2 contains X and the problem is completely solved.

Second solution, using Pascal's theorem. Assume that the line A_1A_2 intersect the circumcircle of the trianlge ABC at A_2 and X. Let $XB_2 \cap AC = B_1'$. Let us apply the Pascal's theorem on the points A, B, C, A_2, B_2, X according the diagram:



It follows that the points A_1, B_1', and M are colinear. Hence $B_1' \in A_1M$. According to the definition of the point B_1' we have $B_1' \in AC$ hence $B_1' = A_1M \cap AC = B_1$. The conclusion is that the points X, B_1, B_2 are colinear. Analogously we prove that the points X, C_1, C_2 are colinear, hence the lines A_1A_2, B_1B_2, C_1C_2 intersect at X that belongs to the circumcircle of the triangle ABC.

13. It is well known (from the theory of pedal triangles) that pedal triangles corresponding to the isogonally conjugated points have the common circumcircle, so called *pedal circle* of the points P and Q. The center of that circle which is at the same time the midpoint of PQ will be denoted by R. Let $P_1' = PP_1 \cap Q_1R$ and $P_2' = PP_2 \cap Q_2R$ (the points P_1' and P_2' belong to the pedal circle of the point P, as point on the same diameters as Q_1 and Q_2 respectively). Using the Pascal's theorem on the points Q_1, P_2, P_2', Q_2, P_1, P_1' in the order shown by the diagram



we get that the points P, R, X_1 are colinear or $X_1 \in PQ$. Analogously the points X_2, X_3 belong to the line PQ.

14. Let us recall the statement according to which the circle l is invariant under the inversion with respect to the circle k if and only if $l = k$ or $l \perp k$.

    Since the point M belongs to the polar of the point A with respect to k we have $\angle MA^*A = 90^\circ$ where $A^* = \psi_l(A)$. Therefore $A^* \in l$ where l is the circle with the radius AM. Analogously $M^* \in l$. However from $A \in l$ we get $A^* \in l^*$; $A^* \in l$ yields $A \in l^*$ (the inversion is inverse to itself) hence $\psi_l(A^*) = A$). Similarly we get $M \in l^*$ and $M^* \in l^*$. Notice that the circles l and l^* have the four common points A, A^*, M, M^*, which is exactly two too much. Hence $l = l^*$ and according to the statement mentioned at the beginning we conclude $l = k$ or $l \perp k$. The case $l = k$ can be easily eliminated, because the circle l has the diameter AM, and AM can't be the diameter of k because A and M are conjugated to each other.

    Thus $l \perp k$, q.e.d.

15. Let $J = KL \cap MN$, $R = l \cap MN$, $X_\infty = l \cap AM$. Since MN is the polar of A from $J \in MN$ we get $\mathscr{H}(K,L;J,A)$. From $KLJA \overset{M}{\underset{\overline{\wedge}}{=}} PQRX_\infty$ we also have $\mathscr{H}(P,Q;R,X_\infty)$. This implies that R is the midpoint of PQ.

16. Let Q be the intersection point of the lines AP and BC. Let Q' be the point of BC such that the ray AQ' is isogonal to the ray AQ in the triangle ABC. This exactly means that $\angle Q'AC = \angle BAQ$ i $\angle BAQ' = \angle QAC$.

    For an arbitrary point X of the segment BC, the sine theorem applied to triangles BAX and XAC yields
    $$\frac{BX}{XC} = \frac{BX}{AX}\frac{AX}{XC} = \frac{\sin\angle BAX}{\sin\angle ABX}\frac{\sin\angle ACX}{\sin\angle XAC} = \frac{\sin\angle ACX}{\sin\angle ABX}\frac{\sin\angle BAX}{\sin\angle XAC} = \frac{AB}{AC}\frac{\sin\angle BAX}{\sin\angle XAC}.$$
    Applying this to $X = Q$ and $X = Q'$ and multiplying together afterwards we get
    $$\frac{BQ}{QC}\frac{BQ'}{Q'C} = \frac{AB}{AC}\frac{\sin\angle BAQ}{\sin\angle QAC}\frac{AB}{AC}\frac{\sin\angle BAQ'}{\sin\angle Q'AC} = \frac{AB^2}{AC^2}. \tag{9}$$
    Hence if we prove $BQ/QC = AB^2/AC^2$ we would immediately have $BQ'/Q'C = 1$, making Q' the midpoint of BC. Then the line AQ is isogonaly conjugated to the median, implying the required statement.

    Since P belongs to the polars of B and C, then the points B and C belong to the polar of the point P, and we conclude that the polar of P is precisely BC. Consider the intersection D of the line BC with the tangent to the circumcircle at A. Since the point D belongs to the polars of A and P, AP has to be the polar of D. Hence $\mathscr{H}(B,C;D,Q)$. Let us now calculate the ratio BD/DC. Since the triangles ABD and CAD are similar we have $BD/AD = AD/CD = AB/AC$. This implies $BD/CD = (BD/AD)(AD/CD) = AB^2/AC^2$. The relation $\mathscr{H}(B,C;D,Q)$ implies $BQ/QC = BD/DC = AB^2/AC^2$, which proves the statement.

17. The statement follows from the Brianchon's theorem applied to $APBMCN$.

18. Applying the Brianchon's theorem to the hexagon $AMBCPD$ we get that the line MP contains the intersection of AB and CD. Analogously, applying the Brianchon's theorem to $ABNCDQ$ we get that NQ contains the same point.

19. The Brokard's theorem claims that the polar of $F = AD \cap BC$ is the line $f = EO$. Since the polar of the point on the circle is equal to the tangent at that point we know that $K = a \cap d$, where a and d are polars of the points A and D. Thus $k = AD$. Since $F \in AD = k$, we have $K \in f$ as well. Analogously we can prove that $L \in f$, hence the points E, O, K, L all belong to f.

20. Let $G = AD \cap BC$. Let k be the circumcircle of $ABCD$. Denote by k_1 and k_2 respectively the circumcircles of $\triangle ADF$ and $\triangle BCF$. Notice that AD is the radical axis of the circles k and k_1; BC the radical axis of k and k_2; and FH the radical axis of k_1 and k_2. According to the famous theorem these three radical axes intersect at one point G. In other words we have shown that the points F, G, H are colinear.

    Without loss of generality assume that F is between G and H (alternatively, we could use the oriented angles). Using the inscribed quadrilaterals $ADFH$ and $BCFH$, we get $\angle DHF = \angle DAF = \angle DAC$ and $\angle FHC = \angle FBC = \angle DBC$, hence $\angle DHC = \angle DHF + \angle FHC = \angle DAC + \angle DBC = 2\angle DAC = \angle DOC$. Thus the points D, C, H, and O lie on a circle. Similarly we prove that the points A, B, H, O lie on a circle.

    Denote by k_3 and k_4 respectively the circles circumscribed about the quadrilaterals $ABHO$ and $DCHO$. Notice that the line AB is the radical axis of the circles k and k_3. Simlarly CD and OH, respectively, are those of the pairs of circles (k, k_2) and (k_3, k_4). Thus these lines have to intersect at one point, and that has to be E. This proves that the points O, H, and E are colinear.

    According to the Brocard's theorem we have $FH \perp OE$, which according to $FH = GH$ and $OE = HE$ in turn implies that $GH \perp HE$, q.e.d.