Problem T–1

Determine all pairs of polynomials (P, Q) with real coefficients satisfying

$$P(x + Q(y)) = Q(x + P(y))$$

for all real numbers x and y.

Problem T–2

Determine the smallest possible real constant C such that the inequality

$$|x^3 + y^3 + z^3 + 1| \leqslant C|x^5 + y^5 + z^5 + 1|$$

holds for all real numbers x, y, z satisfying $x + y + z = -1$.

Problem T–3

There is a lamp on each cell of a 2017×2017 square board. Each lamp is either on or off. A lamp is called *bad* if it has an even number of neighbours that are on. What is the smallest possible number of bad lamps on such a board?

(Two lamps are neighbours if their respective cells share a side.)

Problem T–4

Let $n \geqslant 3$ be an integer. A sequence $P_1, P_2, \ldots, P_n$ of distinct points in the plane is called *good* if no three of them are collinear, the polyline $P_1P_2 \ldots P_n$ is non-self-intersecting and the triangle $P_iP_{i+1}P_{i+2}$ is oriented counterclockwise for every $i = 1, 2, \ldots, n-2$.

For every integer $n \geqslant 3$ determine the greatest possible integer k with the following property: there exist n distinct points $A_1, A_2, \ldots, A_n$ in the plane for which there are k distinct permutations $\sigma\colon \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ such that $A_{\sigma(1)}, A_{\sigma(2)}, \ldots, A_{\sigma(n)}$ is good.

(A polyline $P_1P_2 \ldots P_n$ consists of the segments $P_1P_2, P_2P_3, \ldots, P_{n-1}P_n$.)

Time: 5 hours
Time for questions: 60 min
Each problem is worth 8 points.
The order of the problems does not depend on their difficulty.

Problem T–5

Let ABC be an acute-angled triangle with $AB > AC$ and circumcircle Γ. Let M be the midpoint of the shorter arc BC of Γ, and let D be the intersection of the rays AC and BM. Let $E \neq C$ be the intersection of the internal bisector of the angle ACB and the circumcircle of the triangle BDC. Let us assume that E is inside the triangle ABC and there is an intersection N of the line DE and the circle Γ such that E is the midpoint of the segment DN.

Show that N is the midpoint of the segment $I_B I_C$, where I_B and I_C are the excentres of ABC opposite to B and C, respectively.

Problem T–6

Let ABC be an acute-angled triangle with $AB \neq AC$, circumcentre O and circumcircle Γ. Let the tangents to Γ through B and C meet each other at D, and let the line AO intersect BC at E. Denote the midpoint of BC by M and let AM meet Γ again at $N \neq A$. Finally, let $F \neq A$ be a point on Γ such that A, M, E and F are concyclic. Prove that FN bisects the segment MD.

Problem T–7

Determine all integers $n \geqslant 2$ such that there exists a permutation $x_0, x_1, \ldots, x_{n-1}$ of the numbers $0, 1, \ldots, n-1$ with the property that the n numbers

$$x_0, \quad x_0 + x_1, \quad \ldots, \quad x_0 + x_1 + \ldots + x_{n-1}$$

are pairwise distinct modulo n.

Problem T–8

For an integer $n \geqslant 3$ we define the sequence $\alpha_1, \alpha_2, \ldots, \alpha_k$ as the sequence of exponents in the prime factor decomposition of $n! = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$, where $p_1 < p_2 < \cdots < p_k$ are primes.

Determine all integers $n \geqslant 3$ for which $\alpha_1, \alpha_2, \ldots, \alpha_k$ is a geometric progression.

Time: 5 hours
Time for questions: 60 min
Each problem is worth 8 points.
The order of the problems does not depend on their difficulty.