I–1. Problem

Find all surjective functions $f\colon \mathbb{N} \to \mathbb{N}$ such that for all positive integers a and b, exactly one of the following equations is true:

$$f(a) = f(b),$$
$$f(a+b) = \min\{f(a), f(b)\}.$$

Remarks: $\mathbb{N}$ denotes the set of all positive integers. A function $f\colon X \to Y$ is said to be surjective if for every $y \in Y$ there exists $x \in X$ such that $f(x) = y$.

I–2. Problem

Let $n \geqslant 3$ be an integer. An *inner diagonal* of a *simple n-gon* is a diagonal that is contained in the n-gon. Denote by $D(P)$ the number of all inner diagonals of a simple n-gon P and by $D(n)$ the least possible value of $D(Q)$, where Q is a simple n-gon. Prove that no two inner diagonals of P intersect (except possibly at a common endpoint) if and only if $D(P) = D(n)$.

Remark: A simple n-gon is a non-self-intersecting polygon with n vertices. A polygon is not necessarily convex.

I–3. Problem

Let $ABCD$ be a cyclic quadrilateral. Let E be the intersection of lines parallel to AC and BD passing through points B and A, respectively. The lines EC and ED intersect the circumcircle of AEB again at F and G, respectively. Prove that points C, D, F, and G lie on a circle.

I–4. Problem

Find all pairs of positive integers (m, n) for which there exist relatively prime integers a and b greater than 1 such that

$$\frac{a^m + b^m}{a^n + b^n}$$

is an integer.

Time: 5 hours
Time for questions: 60 min
Each problem is worth 8 points.
The order of the problems does not depend on their difficulty.