Problem T–1

Determine the lowest possible value of the expression

$$\frac{1}{a+x}+\frac{1}{a+y}+\frac{1}{b+x}+\frac{1}{b+y},$$

where a, b, x, and y are positive real numbers satisfying the inequalities

$$\frac{1}{a+x}\geqslant\frac{1}{2},\quad\frac{1}{a+y}\geqslant\frac{1}{2},\quad\frac{1}{b+x}\geqslant\frac{1}{2},\quad\text{and}\ \frac{1}{b+y}\geqslant 1\,.$$

Problem T–2

Determine all functions $f\colon\mathbb{R}\longrightarrow\mathbb{R}$ such that

$$xf(xy)+xyf(x)\geqslant f(x^2)f(y)+x^2y$$

holds for all $x,y\in\mathbb{R}$.

Problem T–3

Let K and L be positive integers. On a board consisting of $2K\times 2L$ unit squares an ant starts in the lower left corner square and walks to the upper right corner square. In each step it goes horizontally or vertically to a neighbouring square. It never visits a square twice. At the end some squares may remain unvisited.
In some cases the collection of all unvisited squares forms a single rectangle. In such cases, we call this rectangle MEMO*rable*.

Determine the number of different MEMOrable rectangles.

Remark. Rectangles are different unless they consist of exactly the same squares.

Problem T–4

In Happy City there are 2014 citizens called $A_1, A_2, \ldots, A_{2014}$. Each of them is either *happy* or *unhappy* at any moment in time. The mood of any citizen A changes (from being unhappy to being happy or vice versa) if and only if some other happy citizen smiles at A. On Monday morning there were N happy citizens in the city.

The following happened on Monday during the day: citizen A_1 smiled at citizen A_2, then A_2 smiled at A_3, etc., and, finally, A_{2013} smiled at A_{2014}. Nobody smiled at anyone else apart from this. Exactly the same repeated on Tuesday, Wednesday and Thursday. There were exactly 2000 happy citizens on Thursday evening.

Determine the largest possible value of N.

Time: 5 hours
Time for questions: 60 min
Each problem is worth 8 points.
The order of the problems does not depend on their difficulty.

Problem T–5

Let ABC be a triangle with $AB < AC$. Its incircle with centre I touches the sides BC, CA, and AB in the points D, E, and F respectively. The angle bisector AI intersects the lines DE and DF in the points X and Y respectively. Let Z be the foot of the altitude through A with respect to BC.

Prove that D is the incentre of the triangle XYZ.

Problem T–6

Let the incircle k of the triangle ABC touch its side BC at D. Let the line AD intersect k at $L \neq D$ and denote the excentre of ABC opposite to A by K. Let M and N be the midpoints of BC and KM respectively.

Prove that the points B, C, N, and L are concyclic.

Problem T–7

A finite set of positive integers A is called *meanly* if for each of its nonempty subsets the arithmetic mean of its elements is also a positive integer. In other words, A is meanly if $\frac{1}{k}(a_1 + \ldots + a_k)$ is an integer whenever $k \geqslant 1$ and $a_1, \ldots, a_k \in A$ are distinct.

Given a positive integer n, determine the least possible sum of the elements of a meanly n-element set.

Problem T–8

Determine all quadruples (x, y, z, t) of positive integers such that

$$20^x + 14^{2y} = (x + 2y + z)^{zt}\,.$$

Time: 5 hours
Time for questions: 60 min
Each problem is worth 8 points.
The order of the problems does not depend on their difficulty.