Problem I–1

Determine all functions $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$xf(y) + f\big(xf(y)\big) - xf\big(f(y)\big) - f(xy) = 2x + f(y) - f(x+y)$$

holds for all $x, y \in \mathbb{R}$.

Problem I–2

We consider dissections of regular n-gons into $n-2$ triangles by $n-3$ diagonals which do not intersect inside the n-gon. A *bicoloured triangulation* is such a dissection of an n-gon in which each triangle is coloured black or white and any two triangles which share an edge have different colours. We call a positive integer $n \geqslant 4$ *triangulable* if every regular n-gon has a bicoloured triangulation such that for each vertex A of the n-gon the number of black triangles of which A is a vertex is greater than the number of white triangles of which A is a vertex.

Find all triangulable numbers.

Problem I–3

Let ABC be a triangle with $AB < AC$ and incentre I. Let E be the point on the side AC such that $AE = AB$. Let G be the point on the line EI such that $\measuredangle IBG = \measuredangle CBA$ and such that E and G lie on opposite sides of I.

Prove that the line AI, the perpendicular to AE at E, and the bisector of the angle $\measuredangle BGI$ are concurrent.

Problem I–4

For integers $n \geqslant k \geqslant 0$ we define the *bibinomial coefficient* $\left(\!\binom{n}{k}\!\right)$ by

$$\left(\!\binom{n}{k}\!\right) = \frac{n!!}{k!!(n-k)!!}\,.$$

Determine all pairs (n, k) of integers with $n \geqslant k \geqslant 0$ such that the corresponding bibinomial coefficient is an integer.

Remark. The double factorial $n!!$ is defined to be the product of all even positive integers up to n if n is even and the product of all odd positive integers up to n if n is odd. So e.g. $0!! = 1$, $4!! = 2 \cdot 4 = 8$, *and* $7!! = 1 \cdot 3 \cdot 5 \cdot 7 = 105$.

Time: 5 hours
Time for questions: 60 min
Each problem is worth 8 points.
The order of the problems does not depend on their difficulty.